Quiz 9.2: Sample Answers

1. Find the intervals of increase and decrease of $f(x)$, where

$$f(x) = 120x + x^3 - 39/2x^2$$

We first find where the derivative equals 0:

$$\begin{aligned} f'(x) = 120 + 3x^2 - 39x &= 0 \\ x^2 - 13x + 40 &= 0 \\ (x-8)(x-5) &= 0 \end{aligned}$$

Thus when $x = 5$ or $x = 8$, $f'(x) = 0$. So the intervals will be $(-\infty, 5), (5, 8), (8, \infty)$. We then take a test point in each interval to see whether the derivative is positive or negative. We'll use the test points 0, 6, and 9. One can check that $f'(0) > 0$, $f'(6) < 0$, and $f'(9) > 0$.

Thus on $(-\infty, 5)$ and $(8, \infty)$, $f(x)$ is increasing, while on $(5, 8)$, $f(x)$ is decreasing.

2. Find the local maximums and minimums of $f(x)$, where

$$f(x) = 90x + x^3 - 33/2x^2 + 30$$

We first find where the derivative equals 0:

$$\begin{aligned} f'(x) = 90 + 3x^2 - 33x &= 0 \\ x^2 - 11x + 30 &= 0 \\ (x-6)(x-5) &= 0 \end{aligned}$$

Thus when $x = 5$ or $x = 6$, $f'(x) = 0$. So the intervals will be $(-\infty, 5), (5, 6), (6, \infty)$. We then take a test point in each interval to see whether the derivative is positive or negative. We'll use the test points 0, 5.5, and 10. One can check that $f'(0) > 0$, $f'(5.5) < 0$, and $f'(10) > 0$.

Thus on $x = 5$ is a local maximum (changes from positive to negative), and $x = 6$ is a local minimum (changes from negative to positive).

3. Find the intervals of concavity of the function $f(x) = 84x + 4x^3 - 33x^2$.

   To find intervals of concavity, we need to find where $f''(x) = 0$. We calculate:

$$f'(x) = 84 + 12x^2 - 66x$$

   and so

$$f''(x) = 24x - 66$$

   Thus $f''(x) = 0$ when $x = 66/24 = 11/4$. So the intervals will be $(-\infty, 11/4)$, $(11/4, \infty)$. We take a test point in each: 0 and 10. It is easy to check that $f''(0) < 0$ while $f''(10) > 0$.

   So $f(x)$ is concave down on $(-\infty, 11/4)$ and concave up on $(11/4, \infty)$.

4. Find the local maximums and minimums of

$$f(x) = \frac{2x^2}{3x^2 - 5x + 2}$$

   We need to find where $f'(x) = 0$:

$$\begin{aligned} f'(x) = \frac{(3x^2 - 5x + 2)(4x) - (6x - 5)(2x^2)}{(3x^2 - 5x + 4)^2} &= 0 \\ \frac{12x^3 - 20x^2 + 8x - 12x^3 + 10x^2}{(3x^2 - 5x + 4)^2} &= 0 \\ -10x^2 + 8x &= 0 \\ -x(10x - 8) &= 0 \end{aligned}$$

   Thus $f'(x) = 0$ when $x = 0$ or $x = 8/10 = 4/5$. Thus the intervals of increase/decrease are $(-\infty, 0), (0, 4/5), (4/5, \infty)$. We take a test point in each interval: -1, 0.5, and 1. We get $f'(-1) < 0$, $f'(0.5) < 0$, and $f'(1) > 0$.

   Thus $x = 0$ is a local minimum, while $x = 4/5$ is a local maximimum.

5. Find the intervals of concavity of

$$f(x) = \frac{x^2}{3x^2 - 3}$$

To find intervals of concavity, we need to find where $f''(x) = 0$. We calculate:

$$\begin{aligned} f'(x) &= \frac{(3x^3 - 3)(2x) - (6x)(x^2)}{(3x^2 - 3)^2} \\ &= \frac{6x^2 - 6x - 6x^2}{(3x^2 - 3)^2} \\ &= \frac{-6x}{(3x^2 - 3)^2} \end{aligned}$$

and so

$$\begin{aligned} f''(x) = \frac{(3x^3 - 3)^2(-6) - 2(3x^2 - 3)(6x)(-6x)}{(3x^2 - 3)^4} &= 0 \\ \frac{(9x^4 - 19x^2 + 9)(-6) + 72x^2(3x2 - 3)}{(3x^2 - 3)^4} &= 0 \\ \frac{(9x^4 - 19x^2 + 9)(-6) + 72x^2(3x2 - 3)}{(3x^2 - 3)^4} &= 0 \\ \frac{-54x^4 + 108x^2 - 54 + 216x^4 - 216x^2}{(3x^2 - 3)^4} &= 0 \\ 162x^4 - 108x^2 - 54 &= 0 \\ 3x^4 - 2x^2 - 1 &= 0 \end{aligned}$$

To solve for x, we then use the quadratic formulta to get that

$$x^2 = \frac{-(-2) \pm \sqrt{2^2 - 4(3)(-1)}}{3(2)} = \frac{2 \pm 4}{6}$$

Thus $x^2 = 1$ or $x^2 = -2/3$. Since x^2 cannot be negative, we only have two solutions: $x = \pm 1$. Thus our intervals of concavity are $(-\infty, -1)$, $(-1, 1)$, and $(1, \infty)$.

We take a test point in each interval: -2, 0, and 2. We find that $f''(-2) > 0$, $f''(0) < 0$, and $f''(2) > 0$. Thus on $(-\infty, -1)$ and $(1, \infty)$ $f(x)$ is concave up, while on $(-1, 1)$, $f(x)$ is concave down.